A case study about the formalization by pupils of a number theory problem

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Abstract. The aim of the paper is to point out how pupils of different ages know how to formalize algebrically a number theory historical problem. The paper was inspired by the new epistemological perspective according to Regis Gras who introduced a new approach with its own characteristics for representing a fuzzy implication. The empirical data were processed using the classical methods of the Statistical Implicative Analysis into a broad range of possible behaviours of pupils regarding a fuzzy a priory-analysis. The obtained results showed a difference between the processing of data by a binary logical assumption in the a priori-analysis and the same ones processed from a fuzzy point of view. This is significative for the evaluation of mathematical skills of pupils at different levels.

1 Introduction

This paper concerns with a didactic research about the skilfulness of formalization by pupils facing a given mathematical problem having a historical background, namely the determination of all the prime numbers which are sums of two squares.

The first formulation of the theorem was expressed by Pierre de Fermat (1601-1665) into a letter to Merin Mersenne (1588-1648) on 25 December 1640 in which we read:

Tout nombre premier, qui sourpasse de l'unité un multiple du quaternaire, est une seul fois la somme de deux quarrés, et une seule fois l'hypoténuse d'un triangle rectangle.

Fermat wrote again on the same subject to Frénicle de Bessy (1605-1675) on 15 June 1641:

La proposition fondamentale des triangles rectangles est que tout nombre premier, qui sourpassede l'unité un multiple de 4, est composé de deux quarrés.

In his *Observations sur Diophante* (edited by his son Samuel in 1670) Fermat supplied a method to determine how many times a given number is the sum of two squares and, hence, the hypotenuse of a Pythagorean triangle. In the same observation we read of a procedure for finding a number that is obtained by adding two squares in an arbitrary number of ways. Therefore, the inspiration to study the sum of two squares is related to the Pythagorean triangles.

After Fermat's formulation, the first demonstration of the theorem was given by Leonhard Euler (1707-1783) who performed some years of work to gain a satisfactory proof of it. Another demonstration of the theorem was given by Carl Friedrich Gauss (1777-1855) in his *Disquisitiones Arithmeticae* (1801) by the theory of binary quadratic forms[1].

The theme of the present research arose in an informal manner during a lesson about the Pythagorean theorem and its converse. First, the consideration of the Pythagorean relation got the teacher to ask pupils to try to determine the triples of natural numbers *a*, *b* and *c* such that $a^2 + b^2 = c^2$, i.e. the so called pythagorean triples. On the other hand, the simple observation of some examples of triples as $5^2 = 3^2 + 4^2$ or $13^2 = 5^2 + 12^2$ got the teacher to ask if a same relation can hold for prime numbers too, namely, when a prime number can be written as a sum of two squares? For example, 5 and 13 are sums of two squares, since

$$5 = 1^2 + 2^2, \quad 13 = 2^2 + 3^2,$$

[1] For an extensive history of the demonstrative methods of the theorem see Bussotti [2000].

while 3 and 19 cannot be written as a sum of two squares, since, for example, one can check, for 19, that none of the differences

$19 – 1^2 = 18$, $19 – 2^2 = 15$, $19 – 3^2 = 10$, or $19 – 4^2 = 3$

is a square.

This question, apparently simple, greatly interested the pupils, therefore the teacher posed a general question: to find the algebraic form of *all* the prime numbers which can be written as a sum of two squares. He wanted to falsify or to validate a basic hypothesis redarding the fact that *pupils are not generally able to formalize a number theory problem*.

In order to quantify pupils's answers to the question, the research was carried out by using two a-priori analyses about the possible behaviours of pupils. The first a-priori analysis labelled pupils's behaviours on the basis of the binary logic, while the second one labelled pupils's behaviours on the basis of fuzzy logic. We are intending the implicated variables as variables-interval to be defined by linguistic-numerical terms according to a fuzzy codification approach, in the sense that the used methodology regarding the analysis of the results has been appropriating chosen because, as for the arithmetic and algebraic thought, it is generally difficult to characterize it in a clear and definite manner only by a dichotomous choise between two values (0-1). So, for each classic field S_i of behaviours we have indicated a sequence of fuzzy indicators F_{ij} corresponding to weighted evaluations regarding the awareness degree by pupils about the algebraic thought. We made a fuzzy codification by different weights on each of the F_{ij}.

The data were processed by the software CHIC in both cases and the results were really different, because, while the first a-priori analysis validated the original hypoyhesis, the second one, falsified it, showing that it needs very attention by a teacher before formulating any judgment about the skilfulness of a pupil.

The paper is an attempt to apply the new epistemological perspective of Regis Gras[2] into a didactic context, because the fuzzy implication is suitable to interpret the results of the investigation.

2 The experimentation

The experimentation was carried out at three different public High Schools in Palermo: "Galileo Galilei"(Scientific High School), "Finocchiaro Aprile" (Psyco Social Pedagogic High School) and "ITC Medi" (Technical Commercial High School).

The choice of these three schools is owed to the fact that different typologies of students give more significancy to the results regarding the passage from the arithmetic thought to the algebraic one.

The classes involved were a first, a second and a third class of the three High School (14-16 years old). The range of the age of the students was chosen to investigate, in a largest possible way, the different behaviour and verbalization of the pupils. Everyone was able to understand the language by which the text of the problem was expressed.

As regards the methodology followed, the questionnaire was distributed at the same time to the three different classes, with a table of the first 500 prime numbers and a table of the first 500 squares. The time available to the students was 120 minutes.

[2] Gras R., Spagnolo F. (2004).

3 The a-priori analyses of pupils's behaviours

Classic analysis	Fuzzy analysis
S0: He/she does not understand the problem	F01: errors in understanding the text; F02: calculations with prime number 2;
S1: He/she goes on by trial and error	F11: he/she goes on by random attempts; F12: he/she calculates only the first "10" cases; F13: he/she goes on in a methodical manner;
S2: He/she goes on by arithmetic calculations	F21: he/she does calculate by using both with compound and prime numbers; F22: he/she does calculate the differences between squares at random; F23: he/she does calculate differences between subsequent squares;
S3: He/she generalizes without using algebra formally (he/she does characterize only some relations among calculations approaching to the use of variables, without distinguishing the different role between a parameter and a variable)	F31: Approaching to the use of variable, she/she does consider some large prime numbers as generalization even if he/she doesn't arrive to a general formalization; F32: Approaching to the use of variable, he/she doesn't consider some large prime numbers, he/she only use the "first" case. He/she doesn't arrive to a general formalization;
S4: He/she generalizes by using algebra (he/she does point out the role of the parameter)	F41: he/she writes some algebraic forms with a parameter; F42: he/she puts the parameter erroneously in the form; F43: he/she does not attribute right values to the parameter; F44: he/she performs right calculations with the parameter;
S5: He /she answer "yes"	F51: he/she answer "Yes" after few attempts; F52: he/she answer "Yes" after a lot of attempts;
S6: He /she answer "no"	F61: he/she answer "No" after few attempts; F62: he/she answer "No" after a lot of attempts;

4 Analysis of data: the different evaluation from the binary logic and the fuzzy point of view

The following graphs we are discussing are the more meaningful ones which we obtained by the experimental phase. We did not insert other graphs because they confirmed all that was deduced from the graphs reported here.

We present two different points of view to “read” the results in terms of the validation of the basic hypothesis: a binary/classical logic point of view and a fuzzy one. The data processed by the software CHIC “give” in the two case studies really different results. As we just said before, while the first a-priori analysis validated the original hypoyhesis, the second one falsify it.

The fuzzy evaluation has been labelled in the following manner corresponding to different “behaviours tipologies”:

	S1: Trial and errors			S3: Generalization without using algebra formally (only some relations among calculations approaching to the use of variables)	
	F11	F12	F13	F31	F32
Student 1	0,00	0,33	0,90	0,00	0,67
Student 2	0,00	0,00	0,50	0,00	0,00
Student 3	0,00	0,00	0,50	0,00	0,00
Student 4	0,00	0,00	0,50	0,33	0,00
Student 5	0,00	0,33	0,50	0,00	0,00
Student 6	0,00	0,33	0,50	0,00	0,00
Student 7	0,00	0,33	0,00	0,00	0,00
Student 8	0,00	0,33	0,50	0,00	0,00
Student 9	0,17	0,00	0,50	0,00	090
Student 10	0,17	0,00	0,00	0,00	0,00
Student 11	0,17	0,00	0,00	0,00	0,00
Student 12	0,00	0,33	0,90	0,00	1,00

As regard the result obtained by pupils attending the first year of High School we refer to the comparison of the results finded at all the three High Schools considerated.

By analizing, at first, the problematic from the binary logic point of view, the following similarities tree is referred to the Scientific High School “Galileo Galilei”. It shows us, as more interesting results, how the strategies S0-S1-S2 relevant for the processes by trial and error adopted by the students, are considereted by the software quite similar to the algebric one (S3) in terms of manipulation of formulas.

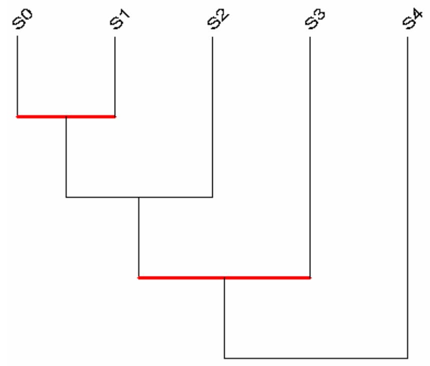


Fig. 1 Binary Similarity tree of Scientific High School "Galileo Galilei"

Perhaps, this interpretation depends on the fact that the *modus operandi* by trials and errors can be considered whether as an arithmetic strategy and as a pure manipulation by letters which has nothing in common with the algebraic formalization. This fact represents for us an open problem which has to be investigated in a future experimentation, even if we have already made a more detailed analysis, by a fuzzy codification.

In fact, a deeper analysis in terms of fuzzy logic shows us how the similarities tree changes radically and "specifies" the precedent graph:

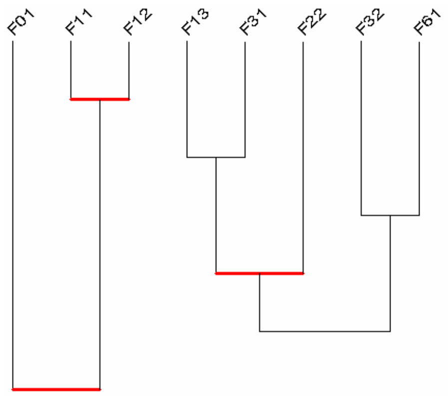


Fig. 2 Fuzzy similarity tree of Scientific High School "Galileo Galilei"

The first significant similarity is the link beetween F11 and F12: the students going on random by trial and error, after few attempts, give up to go on because they are not able to find a general rule and so they stop after few tests.

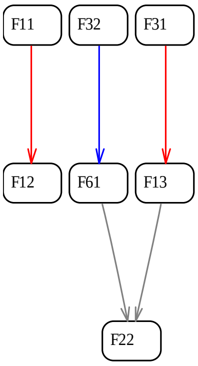
The results finded in the previous graph (S1-S3) are also presented in this *fuzzy similarity tree*, in terms of F13-F31. The students try to arrive at a formalization of the treated problematic but they point out some difficulties to work in a formalizated mileu and so they re-consider again, as a help to their reasoning, the arithmetic strategies. Infact, in the protocols they re-consider some sequence of large prime numbers as single cases of their reasoning. Formalizing equals for them to consider that a property be true when they consider very large natural numbers.

As regard this behavior we could give two different interpretations:

In order to prove the truth of the assertion concerning the posed problem, the student utilizes an "induction process" *sui generis*: i. e. he considers as initial quantities the examples presented in the text of the problem and he thinks that the decomposition of a prime number into the difference between two squares could be true for all the prime numbers if he is able to verify it for some large numbers.

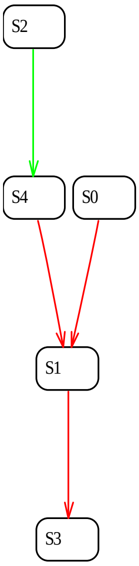
The student goes for abduction (according to Peirce) because he believes that verifing the truth of the assertion for a little set of large prime numbers the truth follows for the whole set of prime numbers.

The link (F13-F31)-F22 shows us that the students are not able to get an overall formalization of the problem because they are linked to a strategy by trail and errors A brief overview of the implicative graph seems to confirm our intepretation of the protocols.



The implications F31 => F13 and F11 =>F12 are valid to the threshold 0.99;
the implication F32 => F61 is instead valid to the threshold 0.95, and the implication F61-f13 => F22 is valid to the threshold 0.60.

The following *implicative binary graph* puts in evidence an interesting element of analysis for the research.



In fact, among the different current implications the S2=>S4 (with threshold 0.99) and S1=>S3 (with threshold 0,90) are relevant.
Notwithstanding it is not easy to argue these implications. In fact the implication between the variables S2 and S4 is not an unexpected result, but how have we interpret it in terms of the algebraic thought?
The S4, even if well defined (*He/she generalizes by using algebra (he/she does point out the role of the parameter*), allows us a wide range for the evaluation, and the codification of the student's behaviour in terms of true/false. For this reason, in our opinion, it does not give an evident result.
Instead, in the graph obtained by interpreting the two variables by the *fuzzy lent*, the S2 and S4 vanish completely underlying our false intuitive reasoning due to the binary analysis.

The implication between the variables S1 and S3 are found again in both the graphs, but if in the binary graph the implication is of the type S1=>S3, in the fuzzy one it is F31=>F13.

So, in our opinion, it means that there is a sort of *bijective per cent relation* between the fields S1 and S3 and inversely between the under fuzzy components F31 and F13.

As regard the Psyco-Social-Pedagogical High School "Finocchiaro Aprile", the *fuzzy similarity tree* points out a link between the strategies F11 and F31. It shows us how the students, testing by trial end error, want to start a sort of generalization of the problem but they point out not a high performance on it, they go on by an incorrect manner.

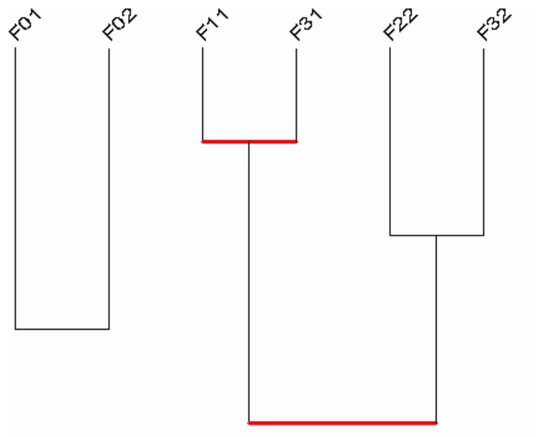


Fig. 3 Fuzzy Similarity tree of High School "Finocchiaro Aprile"

On this subject we consider interesting to underline how a fuzzy analysis allows us to point out and so to evaluate the strategies followed by pupils in a deeper and specific manner. In fact, the analysis of the above *fuzzy similarity tree* points out the presence of a first embryonic algebraic strategy (F31, F32). It is relevant to underline that this strategy had not been pointed out by the first binary analysis. So the fuzzy analysis shows how the basic hypothesis cannot be validated totally while it seemed completely validated by the binary analysis. So by the fuzzy analysis a first algebraic thought is pointed out.

The analysis of the results obtained by the experimentation at the Technical Commercial High School "*Medi*" shows a certain parallelism among the behaviours of pupils in front of the problem. In some ways some obtained results seem unexpected since one waited for a minimun of algebraic thinking compared to the results of the high school "*Finocchiaro Aprile*", seeing that mathematics is not prominent among the disciplines of the course. This is evident from the following fuzzy similarity tree concerning the second class.

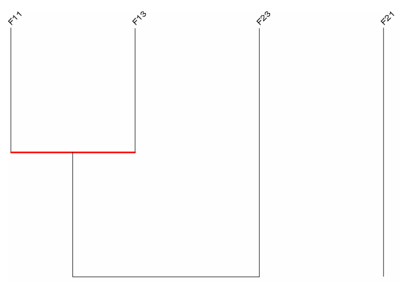
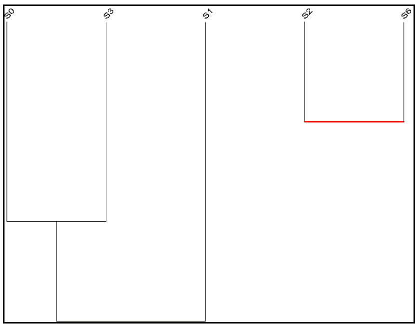
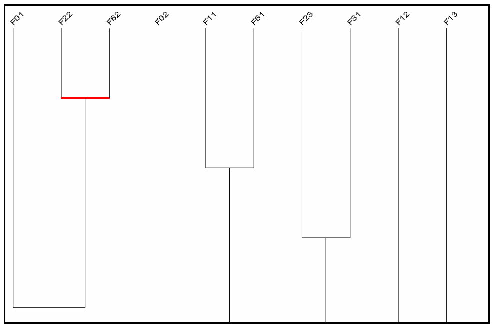


Fig. 4 Fuzzy similarity tree of ITC High School "Medi"

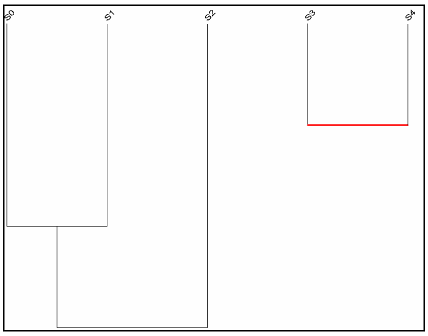
The most interesting results of the experimentation are referred to the data obtained by the third year pupils of High School. So, a comparison among the results obtained at the High Schools "*Galileo Galilei*" and "*Finocchiaro Aprile*" by similarity trees shows not only two different tipologies of curricula but also interesting differences among students's abilities concerning their degree of formalization of a mathematical property, namely their first algebraic competence.



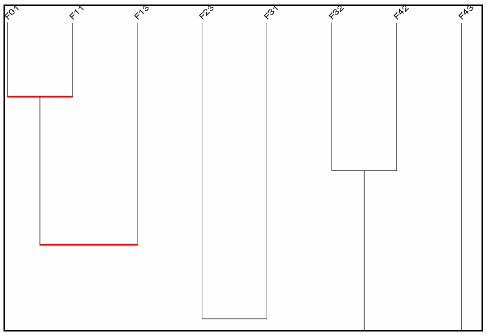
Binary Similarity tree of F. Aprile



Fuzzy Similarity tree of F. Aprile



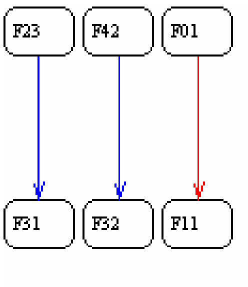
Binary Similarity tree of "G. Galilei"



Fuzzy Similarity tree of "G. Galilei"

So, the similarity trees from the first high school "*Finocchiaro Aprile*" point out not only how the arithmetic reasoning does not carry the required generalization (S2-S6), but also how the arithmetic strategies are completely separated from the algebraic ones. The fuzzy similarity tree allow us to underline how the relation between S2 and S6 of the binary similarity tree is specified in F22-F26. The students, by calculating the differences between squares at random, do not find any general rule even if they make a lot of attempts. Their negative answer to the questions of the test is supported by an adequate number of calculations, which in any case are not sufficient to lead them to a generalization. Another interesting observation can be pointed out from the similarity between F11 and F61. In fact some students go to the same negative answer after having made some calculation at random. aver eseguito qualche calcolo a caso. As we can see, these results are not showed by the binary classical analysis.

The results pointed out by the data collected at the Hight School "*G. Galilei*" are different. The *binary similarity tree* shows a "new" variable which is never present in the other binary similarity trees: the variable referred to the use of the *parameter* and so to a more concret and correct "*use*" of the algebraic thought. The "*binary lent*" points out a deep similarity between S3 and S4, algebraic strategies to solve the problem. The "*fuzzy lent*" allow us to specify, as one deduce from the following implicative graph, this link, in F32 and F42.



Approaching to the use of variable, the students do not consider some large prime numbers as case studies, since they only use the "first" case of their attempts (F42=>F32, implication valid to the threshold 0.90).

They put the parameter but in an erroneous way (in the form), and so they do not arrive to a general formalization.

Other interesting implication could be the F23=>F31 relativa ancora una volta allo stretto legame esistente tra una strategia di stampo aritmetico ed una di tipo pre-algebrico.

5 Open problems

The experimentation, by the fuzzy analysis, pointed out most pupils, except only two into a sample of almost 100 pupils, were not able to formalize a problem concerning the elementary theory of numbers since they principally were not accustomed to using the notions of *variable* and *parameter*. So, an open problem is for us to accertain till point curricula in high schools succeed in deeping this subject in order to allow pupils to carry out generalizations without to fall into the trap of a small numerical evidence.

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Résumé

Le but de cette recherche est de mettre en évidence de quelle façon les élèves d'âges différents savent formaliser algébriquement un problème de théorie des nombres de source historique. Ce travail a été inspiré par une nouvelle perspective épistémologique due à Régis Gras pour représenter une implication floue. Les données empiriques ont été traitées par la méthode classique de l'Analyse Statistique Implicative tout en jouant sur une grande variabilité des comportements possibles des élèves en liaison avec une analyse a priori également floue. Les résultats obtenus ont montré une différence entre l'examen des données avec la logique binaire dans l'analyse a priori et celle examinée d'un point de vue flou. Ce phénomène est significativement intéressant pour l'évaluation des capacités mathématiques des élèves de niveaux différents.

Sunto

Lo scopo di questa ricerca è di mettere in evidenza in che modo allievi di diversa età sanno come formalizzare algebricamente un problema di teoria dei numeri che ha una rilevanza storica. Questo lavoro è stato inspirato dalla nuova prospettiva epistemologica dovuta a Regis Gras per rappresentare una implicazione fuzzy. I dati empirici sono stati esaminati mediante i metodi classici dell'Analisi Statistica Implicativa in un campo di variabilità grande di possibili comportamenti degli allievi relativamente ad un'analisi a priori di tipo fuzzy. I risultati ottenuti hanno mostrato una differenza tra l'esame dei dati affrontato mediante la logica binaria nell'analisi a priori e quello esaminato da un punto di vista fuzzy. Ciò è significativo per la valutazione delle capacità matematiche degli allievi di livello differente.

Appendix 1

The test for the experimentation was the following:

TEST

Name………………………….. Classroom……………….………

Look at these examples:

$$5 = 1^2 + 2^2$$

$$13 = 2^2 + 3^2$$

$$17 = 1^2 + 4^2$$

As you know 5, 13 and 17 are prime numbers. According to you, is it possible to write all the other prime numbers (except 2) in such a way too? Are you able to find a rule?

Let motivate your answer.

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